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# THE SPATIAL SOLOW MODEL WITH CAPITAL AND LABOR DIFFUSION[1]

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**Abstract:** *Spatial economics has become a prominent part of economic sciences over the past decades. One of its goals is to explain the flow and distribution of production factors in space. One possible way to do that is by employing theoretical economic models, such as the Solow model. This article aims to analyze the impact of including a labor diffusion term in the capital equation upon the steady state stability of the spatial Solow model, thereby bridging a gap in the literature. The results indicate that the diffusion coefficient has a profound effect on stability. Namely, high values of the coefficient can make the model unstable, but only if labor reacts to the density of capital.*

**Keywords:** *Spatial Solow Model, Diffusion, Spatial Economics*

**JEL Classification:** C62, R12

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# 1 Introduction

There have been several 'revolutions' in economic theory as well as paradigm shifts. Some of the most heralded ones include the Marginal Revolution or the Keynesian Revolution, which paved the way for modern micro- and macroeconomics, respectively. However, there have been other major contributions which have shaped the way economics is done.

One such example is the introduction of the Solow growth model by Solow (1956). The Solow model provides a modeling framework in which economists could study long-term factors of economic growth. Furthermore, the model (together with its many extensions) renders it possible to study convergence of economies and enables economists partially to explain the observed differences in the product per capita/per worker between various economies (Acemoglu, 2009). However, the early versions of the model neglected the spatial aspect of economic activities.

The spatial dimension of economic activities gained the attention of economists after Paul Krugman's seminal work on economic geography had been published (see Krugman, 1991). Soon after that, general equilibrium models started to emerge which tried to explain the mobility of factors of production as well as the formation of regional structures such as agglomerations (see, for instance, Masahisa, Krugman & Venables, 1999).

A certain line of research within new economic geography strives to explain the distribution as well as mobility (in the following sections, we refer to mobility as diffusion since this term is used in the spatial Solow model literature, stemming from the fact that the underlying equations in the model are of the diffusion type) of both production factors and goods among regions/ spatial units. Better understanding of the underlying forces of factor mobility and distribution can lead to better, tailor-made policies. There are several possible ways to study regional/spatial factor mobility. One such way is to use models with a finite number of regions. Another possibility is to assume that there is a continuum of regions/spatial units. This approach is taken in the spatial Solow model.

Camacho and Zou (2004) try to reconcile the original Solow model which does not take spatial effects into consideration with the economic geography approach by constructing a spatial version of the Solow model. However, the simple version of the spatial Solow model disregards the interplay between capital and labor. This drawback is in part remedied by Juchem Neto and Claeyssen (2014). Nevertheless, their model still ignores the potential effects labor can have on the distribution of capital. Namely, they assume that capital flows from regions with high capital density to regions with low capital density. This assumption is incomplete since capital also flows from regions with low density of labor to regions with high density of labor since the marginal product of capital is higher in those regions.

Therefore, the aim of this article is to analyze the impact of including a labor diffusion term in the capital equation upon the steady state stability of the spatial Solow model. The article is structured as follows: in Section 2 a brief overview of the pertinent literature is provided. In Section 3, we derive and introduce the spatial Solow model. Section 4 summarizes the basic stability results, which are then discussed in Section 5. Conclusions are drawn and prospects for further research are presented in this section as well.

## 2 Literature Review

The pertinent literature on spatial growth models is fairly ample. The very beginnings of spatial-temporal modeling date back to Isard and Liossatos (1975) and Isard (1999). In those articles, the authors advocate for the use of modeling techniques used in natural sciences, namely physics and chemistry, in regional economic modeling. Since the variables of such models depend upon both time and space, the governing equations are no longer ordinary differential equations, which are encountered frequently in economic models, but rather partial differential equations[3], commonly used in natural sciences.

Camacho and Zou (2004) employ these techniques to construct a spatial version of the Solow model. They assume that labor grows exponentially and that capital moves to regions with high marginal productivity. Their results are later utilized in Boucekkine, Camacho and Zou (2009), who provide a spatial version of the Ramsey model originally proposed by Ramsey (1928). However, they exclude the effects of capital and labor interplay.

Juchem Neto and Claeyssen (2014) build upon the spatial Solow model and derive evolution equations for capital and labor making use of the concept

[3] For a detailed treatment, see Evans (2010).

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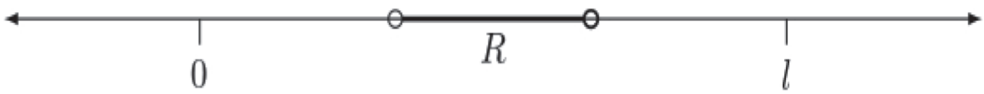
of diffusion. They assume that both the density of capital and labor have an effect on the flow of capital, but only the density of capital impacts the flow of capital. Later they include transport cost in the model (see Juchem Neto, Claeyssen & da Silva Porto Junior, 2014), but the diffusion of capital still depends upon the density of capital only. We strive to bridge this gap in the literature by assuming that the density of labor also impacts the diffusion of capital.

## 3 The Model

We build upon the model proposed by Camacho and Zou (2004), later modified by Juchem Neto and Claeyssen (2014). Let us assume a continuum of local economies denoted by **Ω**. Since we confine the analysis to a 1-dimensional case, we have that **Ω** = ℝ or **Ω** = (0,l),l>0. The boundary of **Ω** is designated by ∂**Ω**. Let us also assume that **Ω** consists of regions denoted by *R*. The space of regions is depicted in Fig. 1.

**Fig. 1:** The Space of Regions




**Source:** author's own calculations

In accordance with the Solow model, we assume that the product *Y(t,x)* (which is now a function of both time *t* and space *x*) is produced through the Cobb-Douglas production function employing capital *K(t,x)* and labor *L(t,x)*. The temporal change of capital in a region *R* is given by

$$\frac{\partial}{\partial t}\int_{R} K(t,x)dx = \int_{R} s\,AK^{\alpha}(t,x)L^{1-\alpha}(t,x) - \delta K(t,x) - \frac{\partial \tau(t,x)}{\partial x}dx \qquad (1)$$

where the last term in the right-hand-side integrand denotes the net rate of capital outflow. For the sake of simplicity, we will drop the argument notation next to the functions and we will denote $sAK^{\alpha}(t,x)L^{1-\alpha}(t,x) - \delta K(t,x)$ as *h*. Economically speaking, Equation (1) expresses that the overall change of capital in a region *R* is equal to the overall investment (which is assumed to increase the capital stock) minus the amount of depreciated capital plus the net rate of capital outflow. Imposing all the necessary mathematical restrictions regarding differentiability upon the functions, we can write

$$\int_R \frac{\partial K}{\partial t} - h + \frac{\partial \tau}{\partial x} dx = 0 \tag{2}$$

Since this equality needs to hold for every region $R$, it follows from the Hahn-Banach Theorem (see, for instance Rudin, 1987) that the following equality needs to hold

$$\frac{\partial K}{\partial t} = h - \frac{\partial \tau}{\partial x} \tag{3}$$

Unlike Juchem Neto and Claeyssen (2014), we assume the following condition for the flux of capital

$$\tau = -d_K \frac{\partial K}{\partial x} + c_K K \frac{\partial L}{\partial x} \tag{4}$$

where $d_K$, $c_K > 0$ are diffusion coefficients. The first term signifies that capital flows from regions of high density of capital to regions with low density. The other term expresses that capital flows to regions with high density of labor. This effect is not taken into consideration in Juchem Neto and Claeyssen (2014), which diminishes the overall realness of the model. Plugging Equation (4) into Equation (3), we obtain the first second-order non-linear partial differential equation (henceforth PDE) of the model

$$\frac{\partial K}{\partial t} = h + d_K \frac{\partial^2 K}{\partial x^2} - c_K \frac{\partial}{\partial x}\left(K \frac{\partial L}{\partial x}\right) \tag{5}$$

This equation describes the evolution of capital in the model. Just like Equation (1), we have an equation which describes the temporal change of labor in a region $R$

$$\frac{\partial}{\partial t}\int_R L dx = \int_R g - \frac{\partial H}{\partial x} dx \tag{6}$$

This equation expresses that the overall change of labor in a region $R$ is equal to the growth rate of labor (here assumed to be logistic, thus $g = aL - bL^2$) and the net outflow of labor. The flux of labor is given by

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$$H = -d_L \frac{\partial L}{\partial x} + c_L L \frac{\partial K}{\partial x} \tag{7}$$

which is symmetric to the flux of capital. Once again, labor is assumed to move from regions with high density of labor to regions with low density of labor and from regions of low density of capital to regions of high density of capital. Following the same arguments as before, we derive the second second-order non-linear PDE of the model describing the evolution of labor

$$\frac{\partial L}{\partial t} = g + d_L \frac{\partial^2 L}{\partial x^2} - c_L \frac{\partial}{\partial x}\left(L \frac{\partial K}{\partial x}\right) \tag{8}$$

Putting the equations together, the model is described by the following reaction-convection-diffusion system of PDEs

$$\begin{aligned}
&\frac{\partial K}{\partial t} = h + d_K \frac{\partial^2 K}{\partial x^2} - c_K \frac{\partial}{\partial x}\left(K \frac{\partial L}{\partial x}\right), && x \in \Omega,\ t > 0 \\
&\frac{\partial L}{\partial t} = g + d_L \frac{\partial^2 L}{\partial x^2} - c_L \frac{\partial}{\partial x}\left(L \frac{\partial K}{\partial x}\right), && x \in \Omega,\ t > 0 \\
&K = K_0,\ L = L_0, && x \in \Omega, t = 0 \\
&\frac{\partial K}{\partial x} = 0,\ \frac{\partial L}{\partial x} = 0, && x \in \partial\Omega,\ t > 0
\end{aligned} \tag{9}$$

The first two equations of the model are the equations derived above. The last two equalities denote the initial and the Neumann boundary conditions, respectively. The initial conditions specify the initial endowment of capital and labor (or the initial densities thereof) at $t = 0$. The boundary condition means that the economy is closed. Therefore, there is no flow of capital and labor between the economy and the rest of the world. This is assumed to abstract from the effects of capital and labor diffusion from abroad.

If $\Omega = \mathbb{R}$, the boundary conditions become $\lim_{|x| \to \infty} \frac{\partial K}{\partial x} = 0 = \lim_{|x| \to \infty} \frac{\partial L}{\partial x}$.

It can be proven in the same way as in Juchem Neto and Claeyssen (2014) that given non-negative initial conditions, the solution stays non-negative. Since the system is non-linear, it is very unlikely that a closed-form solution thereof can be obtained. Therefore, one has to resort to qualitative and/or numerical analysis of the system.

## 4 Linear Stability Analysis

In this section, we analyze the spatio-temporal[4] steady-state stability of System (9). To this end, it is important to calculate the spatio-temporal steady state of the system. Setting all of the derivatives equal to zero and solving the corresponding algebraic system of equations, we obtain the non-trivial steady state in the form[5]

$$(K_e,\ L_e) = \left(\frac{a}{b}\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}, \frac{a}{b}\right) \tag{10}$$

This steady state coincides with the one analyzed in Juchem Neto and Claeyssen (2014). Mathematically, this steady state describes a spatially homogeneous density of labor and capital which does not change in time. Before linearizing the system, it is convenient to normalize the steady-state so that it be centered at (1,1). For this purpose, we make use of the same substitutions as in Juchem Neto and Claeyssen (2014), that is

$$K^* = \frac{K}{K_e}, L^* = \frac{L}{L_e}, t^* = at, x^* = \sqrt{\frac{a}{d_K}}x \tag{11}$$

Thus, System (9) becomes (keeping Ω the same)

$$\begin{aligned}
&\frac{\partial K^*}{\partial t^*} = h^* + \frac{\partial^2 K^*}{\partial x^{*2}} - c\frac{\partial}{\partial x^*}\left(K^*\frac{\partial L^*}{\partial x^*}\right), && x^* \in \Omega, t^* > 0\\
&\frac{\partial L^*}{\partial t^*} = g^* + d\frac{\partial^2 L^*}{\partial x^{*2}} - e\frac{\partial}{\partial x^*}\left(L^*\frac{\partial K^*}{\partial x^*}\right), && x^* \in \Omega, t^* > 0\\
&K^* = K_0^*,\ L^* = L_0^*, && x^* \in \Omega, t^* = 0\\
&\frac{\partial K^*}{\partial x^*} = 0,\ \frac{\partial L^*}{\partial x^*} = 0, && x^* \in \partial\Omega, t^* > 0
\end{aligned} \tag{12}$$

where

$$h^* = \beta\left(K^{*\alpha}L^{*1-\alpha} - K^*\right), \beta = \frac{\delta}{a}, c = \frac{c_K}{d_K}L_e, g^* = L^*(1 - L^*), d = \frac{d_L}{d_K}, e = \frac{c_L}{d_K}K_e.$$

[4] One could also analyze the stability of the stationary solution, just like in Camacho, and Zou (2004). This would be obtained by setting all time derivatives equal to zero. A non-linear system of ordinary differential equations (henceforth ODE) would be obtained.
[5] It ought to be noted here that there exists a non-trivial steady state of labor due to the logistic law used in the derivation of the model.

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For the sake of clarity, we will drop the asterisks from now on. It is evident that after this substitution, the non-trivial steady state is centered at (1,1). We are now ready to linearize this system around the steady state making use of Taylor's theorem (see, for instance, Rudin, 1976). In the matrix notation, the linearized system becomes (bold denotes vectors and matrices)

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{D}\frac{\partial^2 \boldsymbol{u}}{\partial x^2} = \boldsymbol{J}\boldsymbol{u}, x \in \Omega,\ t > 0 \\ &\boldsymbol{u} = \boldsymbol{u_0}, \qquad\qquad x \in \Omega, t\ =\ 0 \\ &\frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{0}, \qquad\qquad x \in \partial\Omega,\ t > 0 \end{aligned} \tag{13}$$

where

$$u = (K^* - 1, L^* - 1)^T,\ \boldsymbol{D} = \begin{pmatrix} \boldsymbol{-1} & c \\ e & -d \end{pmatrix} \text{ and } \boldsymbol{J} = \begin{pmatrix} -\beta(1-\alpha) & \beta(1-\alpha) \\ 0 & -1 \end{pmatrix}$$

is the Jacobian matrix of ($h^*, g^*$ ) evaluated at the steady state. The obtained linear matrix PDE can be solved employing Fourier analysis. If $\Omega = \mathbb{R}$, the Fourier transform can be applied to yield the solution. In case of a finite interval[6], the solution can be expressed in terms of its Fourier series.[7]

**4. 1 $\Omega = \mathbb{R}$**

First, we apply the Fourier transform to System (13)[8]

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} = \boldsymbol{A}\tilde{\boldsymbol{u}} \tag{14}$$

where $A = J + \omega^2 D, \omega \in \mathbb{R}$. We have thus reduced the PDE to an ODE, which can be readily solved. In terms of the matrix exponential[9], the solution is

$$\tilde{\boldsymbol{u}} = e^{At}\tilde{\boldsymbol{u}}_{\boldsymbol{0}} \tag{15}$$

[6] In this case, the solution is periodically extended to the whole real line making use of the boundary conditions.
[7] For the applications of Fourier analysis in differential equations, see Pinkus and Zafrany (1997).
[8] Making use of its linearity as well as the fact that $\mathcal{F}\left[\frac{df}{dx}\right](\omega) = i\omega\mathcal{F}[f(x)](\omega)$.
[9] $e^{\boldsymbol{A}t} = \sum_{i=0}^{\infty}\frac{(At)^i}{i!}$.

The stability of the steady state is given by real parts of the eigenvalues $\lambda$ of **A**. If the real parts are negative, then the steady state is linearly stable. If at least one real part is positive, the steady state is linearly unstable[10]. The characteristic equation of **A** is of the form

$$\lambda^2 + a_1\lambda + a_2 = 0 \tag{16}$$

where

$a_1 = (\beta(1 - \alpha) + 1 + \omega^2 (1 + d))$,

$a_2 = \omega^4 (d - ec) + \omega^2 (1 + \beta(1 - \alpha)(d - e)) + \beta(1 - \alpha)$. The solution is given by

$$\lambda_1, \lambda_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \tag{17}$$

Propositions 1 and 2 summarize the stability results (we assume that $c \neq 0$ since the case when it is zero is analyzed in Juchem Neto and Claeyssen, 2014). Before stating the propositions, let us prove Lemma 1.

**Lemma 1.** The real parts of both of the eigenvalues given by Equation (16) are negative if and only if $a_2 > 0$. If $a_2 < 0$, then one eigenvalue is a positive real number and the other one is a negative real number. If $a_2 = 0$, one eigenvalue is zero and the other one is a negative real number.

*Proof.* Given the assumptions upon the model coefficients, it always holds that $a^1 > 0$. From Vieta's formulas we have that $\lambda_1 + \lambda_2 = -a_1 < 0$ and $\lambda_1 \lambda_2 = a_2$. If $a_2 \leq 0$, it immediately follows that exactly one of the eigenvalues has a non-negative real part. If $a_2 > 0$, it follows from the first Vieta's formula that both real parts must be negative. The other parts of the lemma follow from Vieta's formulas as well. Hence, the proof is complete.

**Proposition 1.** The steady state is linearly stable if either one of these conditions is fulfilled:

i. $e=0$

ii. $d \geq \max\{ec,e\}$

[10] See, for instance, Wiggins (1990).

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*Proof*. If $e = 0$, then it follows directly that $a_2 > 0$. From Lemma 1 we know that in this case, both eigenvalues have a negative real part, which concludes the proof. If $d \geq max\{ec,e\}$, then $a_2 > 0$ since all of the coefficients as well as $\omega^2$ and $\omega^4$ are non-negative and $\beta\,(1 - \alpha) > 0$, which concludes the proof.

Notice that if $c = 0$, $e \neq 0$, the steady state can still be unstable as shown in Juchem Neto and Claeyssen (2014). This comes down to the asymmetry in the Jacobian matrix **J.**

**Proposition 2.** The steady state is linearly unstable if either one of these conditions is fulfilled:

i. $d < ec$

ii. $c < \frac{d}{e} \wedge c > \frac{d}{e} - \frac{(1+\beta(1-\alpha)(d-e))^2}{4e\beta(1-\alpha)} \wedge e > d + \frac{1}{\beta(1-\alpha)}$

*Proof.* If d<ec, then $\lim_{|\omega|\to\infty} a_2 = -\infty$, so $a_2 < 0$ for large enough $|\omega|$.

Hence, one eigenvalue has a positive real part as proven in Lemma 1. In the second case, the minimum of $a_2$ is negative, therefore, $a_2 < 0$ for some $\omega$, which again concludes the proof.

**4. 2 Ω=(0,l)**

In this case, the solutions to System (13) are of the form $\boldsymbol{u} = \boldsymbol{e}^{\lambda t}\,\boldsymbol{v}$, which leads to the following Sturm-Liouville boundary-value problem

$$\begin{aligned} &\lambda \boldsymbol{v} + \boldsymbol{D}\frac{\boldsymbol{d^2v}}{\boldsymbol{dx^2}} = \boldsymbol{Jv}, x \in \Omega \\ &\frac{\boldsymbol{dv}}{\boldsymbol{dx}} = \boldsymbol{0}, \qquad\qquad x \in \partial\Omega \end{aligned} \tag{18}$$

The solution will therefore be the following Fourier series

$$\boldsymbol{u} = \sum_{n=0}^{\infty} \boldsymbol{C_n}\, e^{\lambda_n t} cos\left(\frac{n\pi x}{l}\right) \tag{19}$$

If the conditions of the second part of Proposition 2 are satisfied, then the steady state is linearly unstable for a sufficiently large *l*.

## 5 Concluding Remarks

The aim of this article is to analyze the impact of including a labor diffusion term (coefficient $c_K$ in Equation (4)) in the capital equation upon the steady state stability of the spatial Solow model. As is shown in the previous section, this coefficient has a profound impact upon the steady state stability. Namely, if it is too high (and at the same time the diffusion coefficient $c_L$ is non-zero), the steady state becomes unstable. Economically, this means that if capital reacts too sensitively to the density of labor, it is harder for the economy to reach the homogeneous spatio-temporal steady state. This is due to the fact that there are two driving forces working in the opposite direction. On the one hand, capital moves to regions with lower relative density of capital and higher relative density of labor. On the other hand, an analogous proposition holds true for labor as well. Therefore, higher density of labor attracts more capital, but in turns causes labor to flow out. However, the inflowing capital attracts more labor and repels capital. Hence, it should not come as a surprise that with the right set of parameters, the economy could oscillate around the steady state or even evolve in a more chaotic or divergent manner. As shown in our analysis, it all comes down to the relative values of diffusion parameters.

An intriguing observation is that if the diffusion coefficient which attracts labor to regions with higher density of capital $c_L$ is zero, the model is stable no matter the value of $c_k$ (as long as it is non-negative, as assumed in our analysis), but the same is not true if $c_K = 0$. This asymmetry of the model is caused by the asymmetry of the Jacobian matrix. In the presented model, the growth rate of labor depends solely upon the level of labor[11], while the rate of change of capital is a function of both capital and labor.

[11] The model would become more realistic if we assumed that the growth rate of labor also depends upon the product, and thus both capital and labor.

If labor does not react to the density of capital, its governing equation postulates that its diffusion is only affected by its own density, while capital still moves to regions with high density of labor. However, the marginal productivity of both capital and labor drop with an increasing amount of these factors of production. This only drives capital out of such a region, but not labor since its diffusion is only affected by the density of labor. This means that over time, the distribution of labor becomes more and more uniform. Once it is perfectly uniform, the marginal productivity of capital will depend solely upon the distribution of capital. Therefore, the economy will gradually converge towards its steady state.

If, however, it is capital which does not react to the density of labor, the situation is different. Higher density of capital moves capital out but attracts labor. While higher density of labor does not attract capital from neighboring regions, it does increase the product in the home region, which in turn increases the production of capital. So while the flow of capital affects labor only through the diffusion mechanism, the flow of labor affects the capital not only through the diffusion mechanism, but through the production function as well.

There are quite a few possible extensions of the model. Firstly, although this version of the model is spatial, it is not necessarily regional. The same governing equations might as well describe the evolution of individual economies in the world economy. The major difference between an economy and a region lies in the openness towards other regions and/or economies. While economies are usually more closed in the sense that capital and labor flows are not as fluent across its borders, regions are more open, thereby making the capital and labor flow more fluent. This could be taken into considerations of the diffusion coefficients were functions of space. That way, it would be possible to model regional economies within a national economy and national economies in the world economy in one model. The diffusion coefficients would simply attain lower values near the boundaries of national economies than within national economies. Secondly, the model as presented in this paper assumes that the flow of capital and labor depends solely upon their densities. However, the diffusion process is smoother between regions with highly developed (not only physical) infrastructure. Therefore, the model could be extended by including the infrastructure. In such a model it might be possible to observe other conflicting phenomena - regions with highly developed infrastructure may attract capital and labor from other regions, but they also make it easier for their own capital and labor to move to other regions. Thirdly, one might study the effects of various regional policies upon the model. Having said that, any of these changes would make the model considerably more complex, thereby rendering its further analysis more difficult.

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